

Introductory Lectures on String Theory¹

A.A. Tseytlin²

Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

We give an elementary introduction to classical and quantum bosonic string theory.

1 Introduction

particles $\longrightarrow$ strings (1- d objects)

1. “effective” strings: 1- d vortices, solitons, Abrikosov vortex in superconductors, “cosmic strings” in gauge theory; these are built out of “matter”, have finite “width”, massive excitations (including longitudinal ones).
2. “fundamental” strings: no internal structure (zero “thickness”), admit consistent quantum mechanical and relativistic description.



Fig.1: Open string

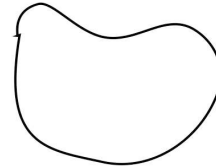


Fig.2: Closed string

Tension = $\frac{\text{mass}}{\text{length}} = T$ - main parameter of fundamental strings.

String vibration modes $\longrightarrow$ quantum particles
discrete spectrum of excitations (∞ number)

$$m^2 = -p^2 + TN, \quad N = 0, 1, 2, \dots, \quad p^2 = -p_0^2 + p_i^2 \quad (1.1)$$

Example: straight open relativistic string rotating about c.o.m.

¹These lectures were prepared in collaboration with Alexei S. Matveev.

²Also at Department of Theoretical Physics, Lebedev Institute, Moscow, Russia



Fig.3: Rotating string

length L

mass $M = T \cdot L$

angular momentum $J \sim P \cdot L$

$P \sim Mc$ (relativistic)

$J \sim P \cdot L \sim M \cdot L \sim T^{-1} M^2$

$M^2 \sim T \cdot J$ property of relativistic string

Angular moment J is quantized in QM

Why strings?

- consistent quantum theory of gravity and other interactions
- effective description of strongly interacting gauge theories

Why not membranes (or p -branes, $p \geq 2$)?

No consistent QM theory of extended objects with $p > 1$ is known

Quantum theory of gravitation:

point-like particle (e.g., graviton): interactions are local $\rightarrow$ UV divergences $\rightarrow$ non-renormalizable (need ∞ number of counterterms)

Graviton scattering amplitudes

UV divergences $\rightarrow$ no consistent description of gravity at short distance

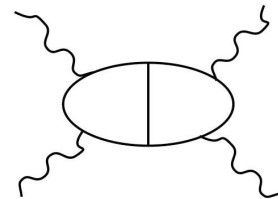


Fig.4: Scattering

String interaction are effectively non-local

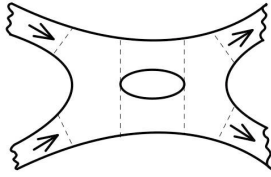


Fig.5: 2- d surfaces as
“world-sheets”

loop amplitudes are finite $\rightarrow$ consistent in QM

graviton and other “light” particles appear as particular string states

UV finiteness: string scale $\sim (\text{tension})^{-1/2}$ plays the role of effective cut-off

Historical origin – in the theory of strong interactions (1968-72)

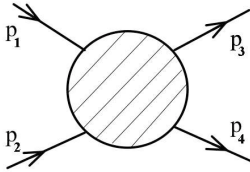


Fig.6: Scattering of
hadrons

$$s = -(p_1 + p_2)^2$$

$$t = -(p_3 + p_4)^2$$

$A(s, t)$ = hadron amplitude

duality observed: $A(s, t) = A(t, s)$

Unusual for field theories

$$\mathcal{L} = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^3$$

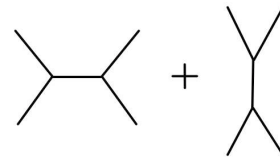


Fig.7: s, t channels are not the
same

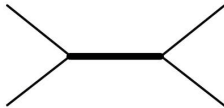


Fig.8: Resonances

But possible in a theory with ∞ number of “resonances” (intermediate states)

$$m^2 \sim n = 1, 2, \dots$$

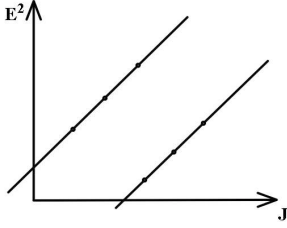


Fig.9: Regge trajectories

Linear relations between energy-squared E^2 of resonances and their spins J characteristic to string theory spectrum were indeed observed in experiments.

2 Classical String Theory

Point-like particle

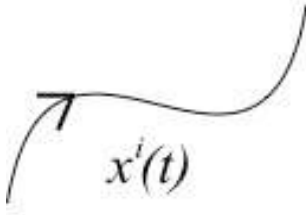


Fig.10: Non-relativistic massive particle

$$\mathcal{L} = \sum_{n=1}^{D-1} \frac{m \dot{x}_n^2}{2} \quad (2.1)$$

Action for a non-relativistic massive particle (assume no higher derivatives)³:

$$S = -mc^2 \int dt \sqrt{1 - \frac{\dot{x}_n^2}{c^2}} \simeq \int dt \left(-mc^2 + \frac{m \dot{x}_n^2}{2} + \dots \right), \quad \dot{x} \ll c \quad (2.2)$$

Momentum:

$$p_n = \frac{\partial \mathcal{L}}{\partial \dot{x}^n} = \frac{m \dot{x}_n}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} \quad (2.3)$$

Invariance of the action: Lorentz transformations

Manifestly relativistic-invariant form (in what follows $c = 1$)

$$S = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}, \quad \mu = (0, n) = (0, 1, \dots, D-1) \quad (2.4)$$

$\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$, D is the dimension of Minkowski space-time $\{x^\mu\}$, τ is a world-line parameter.

Equivalently,

$$S = -m \int ds, \quad (2.5)$$

³In what follows we will omit summation sign so that $\sum_n a_n^2 \equiv a_n^2 \equiv a^2$.

where

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad dx^\mu = \frac{dx^\mu}{d\tau} d\tau$$

$$S = -m \int d\tau \sqrt{(\dot{x}^0)^2 - (\dot{x}^n)^2} \quad (2.6)$$

Symmetries:

1. Space-time: Poincare Group

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu, \quad \Lambda \in SO(1, D-1) \quad (2.7)$$

$$\eta_{\mu\nu} \Lambda^\mu_{\ \mu'} \Lambda^\nu_{\ \nu'} = \eta_{\mu'\nu'} \Rightarrow (x, y) = \eta_{\mu\nu} x^\mu y^\nu = (x', y') \quad (2.8)$$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \text{invariant}$$

2. World-line: reparametrization invariance

$$\tau' = f(\tau), \quad x'^\mu(\tau') = x^\mu(\tau)$$

$$\tau' = \tau + \xi(\tau), \quad x' = x + \delta x \Rightarrow \delta x^\mu = -\xi \dot{x}^\mu$$

“Proper-time” or “static” gauge (special coordinate system in 1- d)

$x^0(\tau) = \tau \equiv t$ — one function is fixed

$x^0(f(\tau)) = x^0(\tau) = \tau \longrightarrow$ fixes $f(\tau)$

(cf. $A_0 = 0$ gauge in Maxwell theory)

$\sqrt{(\dot{x}^0)^2 - (\dot{x}^n)^2} \longrightarrow \sqrt{1 - (\dot{x}^n)^2} = \text{usual relativistic action}$

Action in static gauge

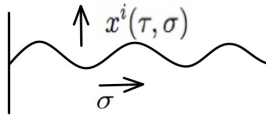
$$S = -m \int dt \sqrt{1 - \dot{x}_n^2} \quad (2.9)$$

Equivalently, changing $\tau \rightarrow x^0(\tau)$

$$S = -m \int dt \sqrt{\left(\frac{dx^0}{d\tau}\right)^2 - \left(\frac{dx^n}{d\tau}\right)^2} = -m \int dx^0 \sqrt{1 - \left(\frac{dx^n}{dx^0}\right)^2} \quad (2.10)$$

2.1 String action

Non-relativistic string



small $\perp$ oscillations ($i = 1, \dots, D-2$)

$$\mathcal{L} \approx \frac{1}{2} T (\dot{x}_i^2 + x_i'^2)$$

$$T - \text{tension} = \frac{\text{mass}}{\text{length}}$$

Fig.11: Non-relativistic string

world surface coordinates $x^i(\tau, \sigma)$ (here $\tau \equiv x^0 = t$)

“transverse” string coordinates $\dot{x}_i = \frac{\partial x_i}{\partial t}$, $x'_i = \frac{\partial x_i}{\partial \sigma}$

$$S \approx \frac{1}{2}T \int d\tau \int d\sigma (\dot{x}_i^2 + x'^2_i) \quad (2.11)$$

Center of mass: $x_i(\tau, \sigma) = \bar{x}_i(\tau) + \dots$

$$S \rightarrow \frac{1}{2}TL \int d\tau \dot{x}_i^2 + \dots = \frac{m}{2} \int d\tau \dot{x}_i^2 + \dots \quad (2.12)$$

Relativistic string

Basic principles:

1. Relativistic invariance (Poincaré group)
2. No internal structure, i.e. no longitudinal oscillations $\rightarrow$ 2- d reparametrization invariance

Relativistic generalization: $x^i \rightarrow x^\mu$

The Nambu-Goto action⁴ is (we assume no higher than first derivatives are present in the action)

$$S = -T \int d\tau \int d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} \quad (2.13)$$

Motion in space-time: $\{x^\mu(\tau, \sigma)\}$, (τ, σ) – world-surface coordinates

$$\dot{x}x' \equiv \dot{x}^\mu x'^\nu \eta_{\mu\nu}, \quad \dot{x}^2 \equiv \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} \quad (2.14)$$

Symmetries:

1. space-time (global) — Poincaré: $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$
2. 2- d world-volume (local) — reparametrizations of world-surface

$$\tau' = f(\tau, \sigma), \quad \sigma' = g(\tau, \sigma)$$

with $x^\mu(\tau, \sigma)$ transforming as scalars in 2- d :

$$x'^\mu(\tau', \sigma') = x^\mu(\tau, \sigma) \quad (2.15)$$

$$\xi^a = (\tau, \sigma), \quad \xi'^a = \xi^a + \zeta^a(\xi)$$

$$x'^\mu(\xi + \zeta) = x^\mu(\xi) \longrightarrow \delta x^\mu = -\zeta^a \partial_a x^\mu, \quad \partial_a = (\partial_0, \partial_1) = (\partial_\tau, \partial_\sigma)$$

Meaning of the action: area of world surface

⁴Y. Nambu, Lectures at the Copenhagen Symposium, 1970; T. Goto, “Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model,” Prog. Theor. Phys. **46**, 1560 (1971).

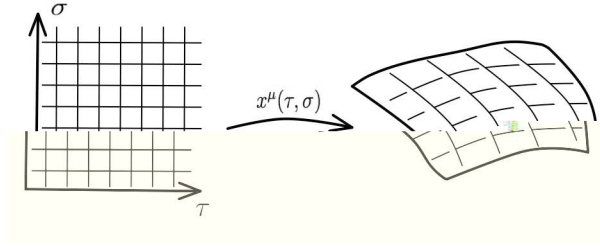


Fig.12: Embedding of (τ, σ) -plane into space-time

Induced metric on world surface ($a, b = 0, 1$)

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = h_{ab}(\xi) d\xi^a d\xi^b \quad (2.16)$$

$$h_{ab} = \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} \quad (2.17)$$

$$h_{ab} = \begin{pmatrix} \dot{x}^2 & \dot{x}x' \\ \dot{x}x' & x'^2 \end{pmatrix}, \quad \det h_{ab} = \dot{x}^2 x'^2 - (\dot{x}x')^2 \quad (2.18)$$

Thus Nambu-Goto action reads

$$S = -T \int d^2\xi \sqrt{-h}, \quad h = \det h_{ab} \quad (2.19)$$

Euclidean signature ($\eta_{\mu\nu} \rightarrow \delta_{\mu\nu}$)

$$\tau \rightarrow i\tau_E, \quad ds^2 = -d\tau^2 + d\sigma^2 \rightarrow d\tau_E^2 + d\sigma^2 \quad (2.20)$$

$$S_E = -iS_M = T \int d\tau_E \int d\sigma \sqrt{\dot{x}^2 x'^2 - (\dot{x}x')^2} \quad (2.21)$$

Element of area:

$$d(\text{Area}) = |\dot{x}||x'| \sin \alpha d\sigma d\tau_E$$

$$\cos \alpha = \frac{\dot{x}x'}{|\dot{x}||x'|}, \quad \sin \alpha = \sqrt{1 - \cos^2 \alpha}$$

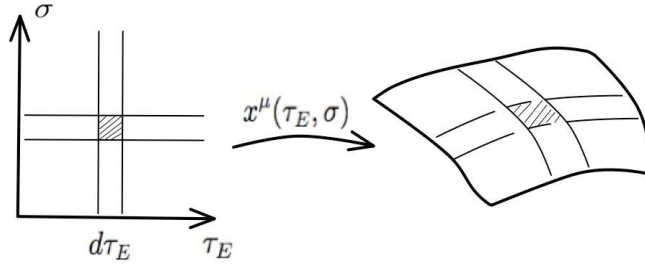


Fig.13: Element of area

$$(\text{Area})_E = \int d\tau_E d\sigma \sqrt{|\dot{x}|^2 |x'|^2} \sqrt{1 - \frac{(\dot{x}x')^2}{|\dot{x}|^2 |x'|^2}} = \int d\tau_E d\sigma \sqrt{\dot{x}^2 x'^2 - (\dot{x}x')^2} \quad (2.22)$$

Static gauge:

$$x^0 = \tau, \quad x^{D-1} = \sigma, \quad x^i(\tau, \sigma) = \text{transverse coordinates}, \quad i = 1, \dots, D-2$$

$$h_{ab} = \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} = \eta_{ab} + \partial_a x^i \partial_b x^i \quad (2.23)$$

Expand action assuming $|\partial_a x^i| \ll 1$ (small oscillations):

$$S = -T \int d^2\xi \sqrt{-\det(\eta_{ab} + \partial_a x^i \partial_b x^i)} \quad (2.24)$$

$$\det \eta_{ab} = -1$$

$$\det(\eta_{ab} + \partial_a x^i \partial_b x^i) = \det \eta_{ab} \det(\delta_d^c + \partial_d x^i \partial_b x^i \eta^{bc})$$

$$\simeq -(1 + \partial_a x^i \partial_b x^i \eta^{ab} + \dots)$$

$$S = -T \int d^2\xi \left(1 + \frac{1}{2} \partial^a x^i \partial_a x^i\right) + \mathcal{O}((\partial x)^4)$$

$$\simeq -m \int d\tau + \frac{1}{2} T \int d\tau \int_0^L d\sigma [(\dot{x}^i)^2 - (x'^i)^2] + \mathcal{O}((\partial x)^4)$$

Equation for small oscillations:

$\ddot{x}_i - x''_i = 0 \longrightarrow$ wave equation in 1-d: transverse waves on the string

2.2 Relation to particle action

Nambu-Goto action describes string as collection of particles moving in direction transverse to the string.

Indeed, use static gauge for τ only ($x^0 = \tau$) and start with:

$$S = -T \int d\tau \int dl \sqrt{1 - V_\perp^2}, \quad (2.25)$$

where

$$dl \equiv d\sigma \sqrt{|x'|^2}, \quad V_\perp^n x'^n = 0 \quad (2.26)$$

and $x \equiv (x^n(\tau, \sigma))$, $n = 1, \dots, D-1$. Solution of $V_\perp^n x'_n = 0$:

$$V_\perp^m = P^{mn} \dot{x}^n, \quad P^{mn} = \delta^{mn} - \frac{x'^m x'^n}{|x'|^2}, \quad P^{mn} x'^n = 0 \quad (2.27)$$

$$V_\perp^2 \equiv V_\perp^n V_\perp^n = P^{nm} P^{nk} \dot{x}^m \dot{x}^k = \dot{x}^2 - \frac{(\dot{x}x')^2}{x'^2} \quad (2.28)$$

$$S = -T \int d\tau \int dl \sqrt{1 - V_{\perp}^2} = -T \int d\tau \int d\sigma \sqrt{x'^2 + (\dot{x}x')^2 - \dot{x}^2 x'^2} \quad (2.29)$$

This action (in $x^0 = \tau$ gauge) is invariant under $\sigma \rightarrow f(\sigma)$.

This coincides with the Nambu-Goto action in the “incomplete” static gauge $x^0 = \tau$:

$$-\det h_{ab} = -(\dot{x}_{\mu} \dot{x}^{\mu})(x'^{\nu} x'_{\nu}) + (\dot{x}^{\mu} x'_{\mu})^2 = (x'_n)^2 - (x'_n)^2 (\dot{x}_m)^2 + (\dot{x}_n x'_n)^2$$

The Nambu-Goto action thus describes a collection of particles “coupled” by the constraint that they should move transversely to the profile of the string (i.e. there should be no longitudinal motions).

2.3 p -brane action

- $p = 0$: particle
- $p = 1$: string
- $p = 2$: membrane (2-brane), etc.

p -brane — $(p + 1)$ -dim world surface Σ^{p+1} embedded in Minkowski space $\mathbb{M}^D$

Embedding: $x^{\mu}(\xi^a)$, $\mu = 1, \dots, D - 1$, $a = 0, \dots, p$

Principles:

1. Poincare invariance (global)
2. world-volume reparametrization invariance (local)
3. no higher derivative (“acceleration”) terms

$$\mathcal{L} \simeq -\frac{1}{2} T_p \partial_a x^i \partial_b x^i \eta^{ab} + \dots$$

where $\eta^{ab} = (-1, +1, \dots, +1)$

Static gauge:

$$x^0 = \xi^0, \dots, x^p = \xi^p; \quad x^i(\xi) = x^i(\xi^0, \dots, \xi^p) \quad (2.30)$$

$x^i(\xi)$ — dynamical (transverse) coordinates

Action that satisfies reparametrization invariance condition has geometric interpretation: volume of induced metric on Σ^{p+1}

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = h_{ab}(x(\xi)) d\xi^a d\xi^b, \quad h_{ab} = \partial_a x^{\mu} \partial_b x^{\nu} \eta_{\mu\nu} \quad (2.31)$$

$$S_p = -T_p \int d^{p+1} \xi \sqrt{-h}, \quad h = \det h_{ab} \quad (2.32)$$

Poincaré and reparametrization invariance are obvious.

Static gauge: use first $(p+1)$ coordinates $x^0, \dots, x^p$ to parametrise the surface

$$x^a = \xi^a, \quad x^i \equiv (\tilde{x}^1, \dots, \tilde{x}^{D-1-p}) \equiv (x^{p+1}, \dots, x^{D-1})$$

$$h_{ab} = \eta_{ab} + \partial_a x^i \partial_b x^i$$

$$\det h_{ab} = -(1 + \eta^{ab} \partial_a x^i \partial_b x^i + \dots)$$

$$S_p = -T_p \int d^{p+1} \xi \sqrt{1 + \partial^a x^i \partial_a x^i + \dots} = -T_p \int d^{p+1} \xi (1 + \frac{1}{2} \partial^a x^i \partial_a x^i + \dots) \quad (2.33)$$

For $p = 0, 1$ one can find a gauge in which the equations of motion become linear; but this is not possible for $p \geq 2$, i.e. the equations are always nonlinear.

Variational principle: $x^\mu \rightarrow x^\mu + \delta x^\mu$

$$\delta S_p = -T_p \int d^{p+1} \xi \delta \sqrt{-\det h_{ab}(x(\xi))} \quad (2.34)$$

$$\delta h_{ab} = \eta_{\mu\nu} \partial_a x^\mu \partial_b (\delta x^\nu) + (a \leftrightarrow b), \quad \delta \sqrt{-h} = \frac{1}{2} \sqrt{-h} h^{ab} \delta h_{ab} \quad (2.35)$$

Used that $\det A = e^{Tr \ln A}$. Then

$$\delta S_p = T_p \int d^{p+1} \xi \delta x^\mu \partial_b (\sqrt{-h} h^{ab} \partial_a x_\mu) - T_p \int d^{p+1} \xi \partial_b (\sqrt{-h} h^{ab} \partial_a x_\mu \delta x^\mu) \quad (2.36)$$

Assume boundary conditions that the boundary term here vanishes; e.g., for $p = 1$ ($0 \leq \sigma \leq L$):

- closed string: $x^\mu(\sigma) = x^\mu(\sigma + L)$, with the boundary condition $\delta x^\mu(\tau_{in}) = \delta x^\mu(\tau_{fin}) = 0$
- open string: free ends – Neumann boundary condition – $\partial_\sigma x^\mu = 0$ at $\sigma = 0, L$

Equation of motion:

$$\partial_b (\sqrt{-h} h^{ab} \partial_a x^\mu) = 0 \quad (2.37)$$

Highly non-linear; simplifies in a special gauge to a linear one for $p = 0, 1$ only.

2.4 Action with auxiliary metric on the world surface

In addition to $x^\mu(\xi)$, let us introduce new auxiliary field $g_{ab}(\xi)$ — metric tensor on Σ^{p+1} .

Classically equivalent action for p -brane is then:

$$I_p(x, g) = -\frac{1}{2}T_p \int d^{p+1}\xi \sqrt{-g} (g^{ab} \partial_a x^\mu \partial_b x_\mu - c) , \quad c = p - 1 \quad (2.38)$$

Action is now quadratic in x^μ but it contains independent $g_{ab}(\xi)$ field. We shall call this “auxiliary metric action”.⁵ Eliminating g_{ab} through its equations of motion yields the same equations for x^μ as following from (2.32).

Interpretation: action for scalar fields $\{x^\mu\}$ in $d = p + 1$ dimensional space Σ^d with metric $g_{ab}(\xi)$.

Variation of (2.38) with respect to x^μ yields

$$\partial_a (\sqrt{-g} g^{ab} \partial_b x^\mu) = 0, \quad (2.39)$$

which is of the same form as for a set of scalar fields x^μ of zero mass in curved space with metric g_{ab} .

Recall that variations of the inverse of the metric tensor and of the determinant of the metric are

$$\delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd}, \quad \delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{ab} \delta g_{ab}. \quad (2.40)$$

With the help of (2.40) the variation of (2.38) with respect to g_{ab} reads

$$\int d^{p+1}\xi \sqrt{-g} \delta g_{ab} \left(\frac{1}{2} g^{ab} (g^{cd} \partial_c x^\mu \partial_d x_\mu - c) - g^{ac} g^{bd} \partial_c x^\mu \partial_d x_\mu \right) = 0. \quad (2.41)$$

Using the notation $h_{ab} = \partial_a x^\mu \partial_b x_\mu$ and lowering the indices with g_{ab} we obtain

$$\frac{1}{2} g_{ab} (g^{cd} \partial_c x^\mu \partial_d x_\mu - c) = h_{ab} , \quad (2.42)$$

⁵In the string ($p = 1$, $c = 0$) case this action originally appeared in the papers [S. Deser and B. Zumino, “A complete action for the spinning string,” Phys. Lett. B **65**, 369 (1976)] and [L. Brink, P. Di Vecchia and P. S. Howe, “A locally supersymmetric and reparametrization invariant action for the spinning string,” Phys. Lett. B **65**, 471 (1976)] which generalized a similar construction for the (super) particle case ($p = 0$, $c = -1$) in [L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. S. Howe, “Local supersymmetry for spinning particles,” Phys. Lett. B **64**, 435 (1976)]. An equivalent action with independent Lagrange multiplier fields that led to the correct constraints (but did not have an immediate geometrical interpretation) appeared earlier in [P.A. Collins and R.W. Tucker, “An action principle and canonical formalism for the Neveu-Schwarz-Ramond string,” Phys. Lett. B **64**, 207 (1976)] (see footnote in the Brink et al paper and the discussion at the end of the Deser and Zumino paper). This “auxiliary metric” form of the classical string action was widely used after Polyakov have chosen it as a starting point for path integral quantization of the string [A. M. Polyakov, “Quantum geometry of bosonic strings,” Phys. Lett. B **103**, 207 (1981)]

i.e.

$$g_{ab} = \lambda h_{ab} , \quad \lambda^{-1} = \frac{1}{2}(g^{ab}h_{ab} - c). \quad (2.43)$$

On the other hand,

$$g^{ab}h_{ab} = \lambda^{-1}g^{ab}g_{ab} = \lambda^{-1}(p+1) , \quad (2.44)$$

i.e.

$$c = \lambda^{-1}(p-1). \quad (2.45)$$

Thus, if $p \neq 1$ and $c = p-1$ we get $\lambda = 1$, or $g_{ab} = h_{ab}$. This means that on the equations of motion the independent metric g_{ab} coincides with the induced metric (for $p = 1$ it is proportional to it up to an arbitrary factor).

Then equation for x^μ in (2.37) becomes

$$\partial_a(\sqrt{-h}h^{ab}\partial_b x^\mu) = 0, \quad (2.46)$$

which is the same as following from the original $S_p[x]$ action (2.32).

Thus, $I_p[x, g]$ and $S_p[x]$ give the same equations of motion and also are equal if we eliminate g_{ab} using the equation of motion $g_{ab} = h_{ab}$

$$\begin{aligned} I_p[x, g]|_{g_{ab}=h_{ab}} &= -\frac{1}{2}T_p \int d^{p+1}\xi \sqrt{-h}(h^{ab}\partial_a x^\mu \partial_b x_\mu - c) \\ &= -\frac{1}{2}T_p \int d^{p+1}\xi \sqrt{-h}(h^{ab}h_{ab} - p + 1) \\ &= -\frac{1}{2}T_p \int d^{p+1}\xi \sqrt{-h} = S_p[x]. \end{aligned}$$

2.5 Special cases

The tensor g_{ab} is symmetric, so it has $(p+1) \times (p+1)$ components which are functions of $(p+1)$ arguments ξ^a . Reparametrization invariance (gauge freedom) is described with $(p+1)$ functions $\xi'^a = f^a(\xi)$. Thus the number of non-trivial components of g_{ab} equals to $\frac{p(p+1)}{2}$.

2.5.1 $p = 0$: particle

In this case $a, b = 0$ and we have only one metric component $g_{ab} = g_{00} = -e^2$. Using the following notation $\xi^a = \tau$, $T_0 = m$, $e = e(\tau)$ we can rewrite (2.38) in the form

$$I_0[e, x] = -\frac{1}{2}T_0 \int d\tau e (-e^{-2}\dot{x}^\mu \dot{x}_\mu + 1) = \frac{1}{2}m \int d\tau (e^{-1}\dot{x}^2 - e) \quad (2.47)$$

Variation with respect to e gives

$$e^{-2}\dot{x}^2 + 1 = 0 , \quad \text{i.e.} \quad e = \sqrt{-\dot{x}^2} .$$

Then

$$I_0|_{e=\sqrt{-\dot{x}^2}} = -m \int d\tau \sqrt{-\dot{x}^2} = S_0[x]. \quad (2.48)$$

Rescaling e by m , i.e. $e(\tau) = m\varepsilon(\tau)$, leads to an action that admits a regular massless limit $m \rightarrow 0$:

$$I_0 = \frac{1}{2} \int d\tau (\varepsilon^{-1} \dot{x}^2 - m^2 \varepsilon). \quad (2.49)$$

The limit $m = 0$ gives an action for the massless relativistic particle

$$I_0^{(0)}[\varepsilon, x] = \frac{1}{2} \int d\tau \varepsilon^{-1} \dot{x}^2. \quad (2.50)$$

Variation of (2.49) with respect to x^μ and ε gives the equations of motion

$$\frac{d}{d\tau} (\varepsilon^{-1} \dot{x}^\mu) = 0 \quad (2.51)$$

$$\dot{x}^\mu \dot{x}_\mu = -m^2 \varepsilon^2. \quad (2.52)$$

Here the number of gauge parameters equals to 1 and is the same as number of components of g_{ab} . Thus $g_{00} = -m^2 \varepsilon^2$ can be completely gauged away. Choosing the special reparametrization gauge

$$\varepsilon(\tau) = 1$$

gives

$$\ddot{x}^\mu = 0 \quad \rightarrow \quad x^\mu = x_0^\mu + p^\mu \tau, \quad (2.53)$$

and

$$\dot{x}^\mu \dot{x}_\mu = -m^2 \quad \rightarrow \quad p^\mu p_\mu = -m^2. \quad (2.54)$$

These are the usual relativistic particle relations.

2.5.2 $p = 1$: string

This is also a special case because $(p-1)\lambda^{-1} = c = 0$ is satisfied identically for any λ . Thus, solution is $g_{ab} = \lambda h_{ab}$, where $\lambda = \lambda(\xi)$ is an arbitrary function of ξ .

The equations for x^μ are still the same for the actions S_p and I_p since

$$\sqrt{-g} g^{ab} = \sqrt{-h} h^{ab}. \quad (2.55)$$

This is due to an extra symmetry that appears in this case. Indeed, in 2 dimensions ($d = p + 1 = 2$) the action ($p = 1, c = 0, T = T_1$)

$$I_1[x, g] = -\frac{1}{2} T \int d^2 \xi \sqrt{-g} g^{ab} \partial_a x^\mu \partial_b x_\mu \quad (2.56)$$

is invariant under the Weyl (or “conformal”) transformations

$$g'_{ab} = f(\xi)g_{ab}. \quad (2.57)$$

This symmetry is present only for massless scalars in $d = 2 = p + 1$. For any p

$$\sqrt{-g'}g'^{ab} = f^{\frac{d-2}{2}}\sqrt{-g}g^{ab}. \quad (2.58)$$

This is another local (gauge) symmetry in addition to the reparametrization invariance. g_{ab} has 3 components – same as the number of gauge parameters of the Weyl (one) and the reparametrization invariance (two) transformations. This allows one to gauge away g_{ab} completely, i.e. to set $g_{ab} \sim \eta_{ab}$, and then obtain a free action for x^μ .

For general p the equation for g_{ab} is the vanishing of the total energy-momentum tensor

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta I_p}{\delta g^{ab}} = T_p(\partial_a x^\mu \partial_b x_\mu - \frac{1}{2}g_{ab}g^{cd}\partial_c x^\mu \partial_d x_\mu) - \frac{c}{2}T_p g_{ab}. \quad (2.59)$$

For $c = 0$ and $g_{ab} = \lambda h_{ab}$ we have $T_{ab}(x) = T(\partial_a x^\mu \partial_b x_\mu - \frac{1}{2}h_{ab}h^{cd}\partial_c x^\mu \partial_d x_\mu) = 0$.

To summarize, in the $p = 1$ case we have the following local gauge transformations

1. Reparametrization invariance:

$$\xi'^a = f^a(\xi) \quad \rightarrow \quad x'^\mu(\xi') = x^\mu(\xi), \quad g'_{ab}(\xi') = \frac{\partial \xi^c}{\partial \xi'^a} \frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi)$$

2. Weyl invariance:

$$x'^\mu(\xi) = x^\mu(\xi), \quad g'_{ab}(\xi) = \sigma(\xi)g_{ab}$$

Total number of gauge functions equals to the number of components of g_{ab} so that it can be gauged away completely.

Namely, one can choose the “conformal” (or “orthogonal”) gauge $g_{ab} = \eta_{ab}$. It is sufficient even to choose the reparametrization gauge only on the Weyl-invariant combination: $\sqrt{-g}g^{ab} = \eta^{ab}$.

Then the equation for x^μ becomes linear. Indeed, consider the reparametrization gauge $g_{ab} = f(\xi)\eta_{ab}$, where f is arbitrary. Then, $g_{00} = -g_{11}$ and $g_{01} = 0$ and from the equation of motion

$$\partial_a(\sqrt{-g}g^{ab}\partial_b x^\mu) = 0 \quad (2.60)$$

we have

$$\partial^a \partial_a x^\mu = 0, \quad \text{i.e.} \quad \ddot{x}^\mu - x''^\mu = 0. \quad (2.61)$$

Also we know that $h_{ab} \sim g_{ab} \sim \eta_{ab}$, where $h_{ab} = \partial_a x^\mu \partial_b x_\mu$. Then, $h_{00} = -h_{11}$ and $h_{01} = 0$. In other words, we have in addition the following differential constraints (containing only 1st time derivatives and generalizing $\dot{x}_\mu^2 = 0$ for a massless particle)

$$\dot{x}_\mu^2 + x_\mu'^2 = 0, \quad \dot{x}^\mu x'_\mu = 0. \quad (2.62)$$

2.6 Meaning of the constraints and first-order actions

2.6.1 Particle case

Let us introduce an independent momentum function $p^\mu(\tau)$. Then the action for $\{x^\mu(\tau), p^\mu(\tau), \varepsilon(\tau)\}$ which is equivalent to (2.49) is

$$\hat{I}_0(x, p, \varepsilon) = \int d\tau \left[\dot{x}^\mu p_\mu - \frac{1}{2} \varepsilon (p^2 + m^2) \right], \quad (2.63)$$

where ε is a Lagrange multiplier imposing the constraint $p^2 + m^2 = 0$.

The variation of (2.63) gives

- $\delta x^\mu :$ $\dot{p}_\mu = 0$
- $\delta p^\mu :$ $\dot{x}^\mu = \varepsilon p^\mu$
- $\delta \varepsilon :$ $p^2 + m^2 = 0$

This action is invariant under the reparametrization with $d\tau \varepsilon(\tau) = d\tau' \varepsilon'(\tau')$. Fixing the gauge as $\varepsilon = 1$ we get the usual equations

$$p^\mu = \dot{x}^\mu, \quad \ddot{x} = 0. \quad (2.64)$$

Eliminating p^μ from the action (2.63) yields (2.49), i.e.

$$\hat{I}_0(x, p, \varepsilon) \Big|_{p=\varepsilon^{-1}\dot{x}} = \frac{1}{2} \int d\tau (\varepsilon^{-1} \dot{x}^2 - \varepsilon m^2). \quad (2.65)$$

Thus $g_{00} = -m^2 \varepsilon^2$ plays the role of a Lagrange multiplier.

2.6.2 String case

Similarly, let us introduce the independent momentum field $p^\mu(\tau, \sigma)$ and consider the alternative action 1-st order action for x^μ, p_μ with the conformal-gauge constraints added with the Lagrange multipliers $\varepsilon(\tau, \sigma)$ and $\mu(\tau, \sigma)$

$$\hat{I}_1(x, p, \varepsilon, \mu) = T \int d\tau d\sigma \left[\dot{x}^\nu p_\nu - \frac{1}{2} \varepsilon (p^2 + x'^2) - \mu p_\nu x'^\nu \right] \quad (2.66)$$

The variation of (2.66) gives

- $\delta x^\nu :$ $\dot{p}_\nu - (\varepsilon x'_\nu)' = 0$
- $\delta p^\nu :$ $p_\nu = \varepsilon^{-1} (\dot{x}_\nu - \mu x'_\nu)$

- $\delta\varepsilon :$ $p^2 + x'^2 = 0$
- $\delta\mu :$ $p^\nu x'_\nu = 0$

This action is invariant under the 2-parameter reparametrization symmetry $\xi'_a = f_a(\xi)$, $\xi = \xi(\tau, \sigma)$. So, one can fix the two gauges $\varepsilon = 1$, $\mu = 0$. Then we get

$$p_\mu = \dot{x}_\mu, \quad \ddot{x}_\mu - x''_\mu = 0, \quad \dot{x}^2 + x'^2 = 0, \quad \dot{x}x' = 0, \quad (2.67)$$

i.e. the same set of equations as in the orthogonal gauge.

As in the particle case, the g_{ab} field is related to the Lagrange multipliers ε and μ . Indeed, eliminating p_μ from the action (2.66) we get

$$\hat{I}_1(x, p, \varepsilon, \mu) \Big|_{p=\varepsilon^{-1}(\dot{x}-\mu x')} = \frac{1}{2}T \int d\tau d\sigma [\varepsilon^{-1}(\dot{x} - \mu x')^2 - \varepsilon x'^2]. \quad (2.68)$$

Comparing the integrand here with the one in the action with independent 2d metric

$$-\frac{1}{2}\sqrt{-g}g^{ab}\partial_a x^\mu \partial_b x_\mu = -\frac{1}{2}\sqrt{-g}(g^{00}\dot{x}^2 + g^{11}x'^2 + 2g^{01}\dot{x}x'),$$

we can identify

$$\varepsilon = -\frac{1}{\sqrt{-g}g^{00}}, \quad \mu = -\frac{g^{01}}{g^{00}}. \quad (2.69)$$

Thus one may say that the components of the 2-d metric g_{ab} play the role of the Lagrange multipliers for the two constraints.

The classical string motion described by the Nambu-Goto action may be interpreted as a motion in phase space subject to the two non-linear constraints. Eliminating the momenta p_μ leads to the action with independent 2d metric, while solving also for the Lagrange multipliers brings us back to the Nambu-Goto action.

By analogy with the particle mass shell constraint, the term x'^2 in the first constraint $p^2 + x'^2 = 0$ can be interpreted as an effective particle mass. The second constraint $px' = 0$ says that the string motion is transverse to the profile of the string.

In the gauge $\sqrt{-g}g^{ab} = \eta^{ab}$ the vanishing of the 2-d scalar stress tensor

$$T_{ab} = \partial_a x^\mu \partial_b x_\mu - \frac{1}{2}\eta_{ab}\eta^{cd}\partial_c x^\mu \partial_d x_\mu = 0 \quad (2.70)$$

means that

$$h_{ab} \sim \eta_{ab} \rightarrow h_{00} + h_{11} = 0, \quad h_{01} = 0 \quad (2.71)$$

Thus

$$T_{00} + T_{11} = 0, \quad T_{01} = 0 \rightarrow \dot{x}^2 + x'^2 = 0, \quad \dot{x}x' = 0. \quad (2.72)$$

2.7 String equations in the orthogonal gauge

Let us introduce the “light-cone” parametrization for the world-sheet

$$\xi^\pm = \tau \pm \sigma, \quad (2.73)$$

$$\partial_\tau = \partial_+ + \partial_-, \quad \partial_\sigma = \partial_+ - \partial_- . \quad (2.74)$$

Then we get

$$\partial_+ \partial_- x^\mu = 0 \quad - \text{equation of motion} \quad (2.75)$$

$$\partial_\pm x^\mu \partial_\pm x_\mu = 0 \quad - \text{constraints} \quad (2.76)$$

Consequently, we can easily write the general solution of (2.75) as a sum of the left-moving and right-moving waves with arbitrary profiles

$$x^\mu = f_+^\mu(\xi^+) + f_-^\mu(\xi^-). \quad (2.77)$$

The constraints then have the form

$$f_+^{\prime 2} = 0, \quad f_-^{\prime 2} = 0. \quad (2.78)$$

2.8 String boundary conditions and some simple solutions

In the conformal gauge the string action reads

$$S = -\frac{1}{2}T \int d\tau d\sigma \partial^a x^\mu \partial_a x_\mu. \quad (2.79)$$

Its variation has the form

$$\delta S = -T \int d\tau d\sigma \delta x^\mu (-\partial^2 x_\mu) - T \int d\tau d\sigma \partial^a (\delta x^\mu \partial_a x_\mu), \quad (2.80)$$

where the variations of x^μ at the starting and the ending values of τ are equal to zero, $\delta x(\tau_1, \sigma) = 0$, $\delta x(\tau_2, \sigma) = 0$.

2.8.1 Open strings

If the string is open, the second term vanishes in the two cases:

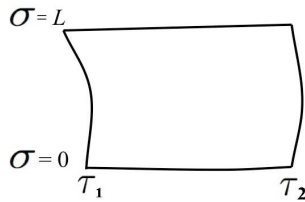


Fig.14

Neumann condition: free ends of the string

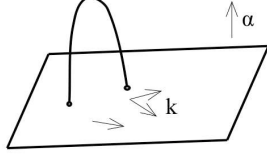
$$\partial_\sigma x^\mu(\tau, \sigma)|_{\sigma=0,L} = 0 \quad (2.81)$$

Dirichlet condition: fixed ends of the string

$$x^\mu(\tau, \sigma)|_{\sigma=0,L} = y_{0,L}^\mu(\tau) \quad (2.82)$$

Here $y_{0,L}^\mu(\tau)$ are some given trajectories. The Dirichlet condition is relevant for the open-string description of D -branes. This condition breaks the Poincaré invariance.

In general, one can also impose mixed boundary conditions. Let us divide the components of x^μ into the two sets



Neumann components (along the brane)

$$\{x^\alpha\} = \{x^0, x^1, \dots, x^p\}$$

Dirichlet components (transverse to brane)

$$\{x^k\} = \{x^{p+1}, x^{p+2}, \dots, x^{D-1}\}$$

Fig.15: Boundary conditions

The boundary conditions read

$$\partial_\sigma x^\alpha|_{\sigma=0,L} = 0, \quad x^k(\tau, \sigma)|_{\sigma=0,L} = y_{0,L}^k(\tau). \quad (2.83)$$

If the space-time contains no Dp -branes, the open strings have free ends in all the directions (Neumann conditions). Then the full Poincaré invariance is unbroken.

2.8.2 Closed strings

For closed strings we impose the periodic condition $x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + L)$ and may choose units so that $L = 2\pi$ and (τ, σ) are dimensionless. Then we get the set of equations

$$\ddot{x} - x'' = 0, \quad \dot{x}^2 + x'^2 = 0, \quad \dot{x}x' = 0 \quad (2.84)$$

with the condition

$$x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + 2\pi). \quad (2.85)$$

The simplest solution is a point-like string

$$x^\mu(\tau, \sigma) = \bar{x}^\mu(\tau) = x_0^\mu + p^\mu \tau, \quad p^\mu p_\mu = 0. \quad (2.86)$$

2.9 Conservation laws

It is useful to split $x^\mu(\tau, \sigma)$ into the coordinate of the center of mass $\bar{x}^\mu(\tau, \sigma)$ and oscillations around it $\tilde{x}^\mu(\tau, \sigma)$

$$x^\mu(\tau, \sigma) = \bar{x}^\mu(\tau) + \tilde{x}^\mu(\tau, \sigma), \quad (2.87)$$

$$\bar{x}^\mu(\tau) = \frac{1}{L} \int d\sigma x^\mu(\tau, \sigma), \quad (2.88)$$

$$\int d\sigma \tilde{x}^\mu(\tau, \sigma) = 0. \quad (2.89)$$

Global symmetries lead via the Noether theorem to quantities that are conserved on the equations of motion.

On the other hand, the local symmetries (reparametrizations and the Weyl transformation) lead to the restrictions on the energy-momentum tensor

$$\partial_a T^{ab} = 0, \quad T_a^a = 0, \quad (2.90)$$

or to the constraints on x^μ after the gauge fixing.

If the action is invariant under some transformation δx^μ , then using Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \partial_a \frac{\partial \mathcal{L}}{\partial \partial_a x^\mu} = 0 \quad (2.91)$$

we get a conserved current

$$j^a = \frac{\partial \mathcal{L}}{\partial \partial_a x^\mu} \delta x^\mu, \quad \partial_a j^a = 0. \quad (2.92)$$

Let $\delta x^\mu = \Lambda^\mu_A \varepsilon^A$, where ε^A are constant parameters. Then $j^a = j_A^a \varepsilon^A$ and $\partial_a j_A^a = 0$. Now it is easy to check that the integral

$$J_A(\tau) = \int d\sigma j_A^0(\tau, \sigma) \quad (2.93)$$

gives a conserved charge

$$\frac{d}{d\tau} J_A(\tau) = 0. \quad (2.94)$$

The string action is invariant under the Poincaré transformations

$$\delta x^\mu = \varepsilon^\mu_\nu x^\nu + \varepsilon^\mu, \quad (2.95)$$

where $\varepsilon^\mu_\nu = -\varepsilon_\nu^\mu$, i.e. $\varepsilon_\mu^\nu \in so(1, D-1)$. Then for space-time translations in the orthogonal gauge $g_{ab} \sim \eta_{ab}$ we get the conserved current (momentum density)

$$p^a = p_\mu^a \varepsilon^\mu = \frac{\partial \mathcal{L}}{\partial \partial_a x^\mu} \varepsilon^\mu = -T \partial^a x_\mu \varepsilon^\mu. \quad (2.96)$$

Recalling that $\dot{x} = \partial_0 x = -\partial^0 x$ and taking (2.87) into account, we get

$$P^\mu = \int d\sigma p^{0\mu} = T \int d\sigma \dot{x}^\mu = T L \dot{\bar{x}}^\mu, \quad (2.97)$$

$$\frac{d}{d\tau} P^\mu = 0. \quad (2.98)$$

Indeed, using the equation of motion and the boundary conditions we get

$$\dot{P}^\mu = T \int d\sigma \ddot{x}^\mu = T \int_0^L d\sigma x''^\mu = T (x'^\mu)|_0^L = 0 \quad (2.99)$$

This conservation law means that there is no momentum flow through boundary.

From (2.97) we get

$$\bar{x}^\mu(\tau) = x_0^\mu + p^\mu \tau, \quad p^\mu = \alpha' P^\mu, \quad TL \equiv \frac{1}{\alpha'} . \quad (2.100)$$

Thus, the center of mass moves with a constant velocity determined by the total momentum of the string.

Remark:

Since the string equation of motion is of the second order formally it has a simple solution linear in both τ and σ

$$x^\mu = x_0^\mu + p^\mu \tau + q^\mu \sigma. \quad (2.101)$$

For the open strings with free ends, i.e. with Neumann boundary conditions we get $q^\mu = 0$. In the case of the closed strings we need $x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + 2\pi)$. A generalized version of this condition can be satisfied if x^μ is a compact (angular) coordinate, e.g.,

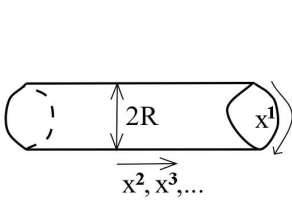


Fig.18: Compactification on cylinder

$$x^1 = R\varphi, \quad \varphi \sim \varphi + 2\pi$$

Then (2.101) is consistent with the “generalized” periodicity

$$x(\tau, \sigma + 2\pi) = x(\tau, \sigma) + 2\pi R w,$$

where $w = 0, 1, 2, \dots$ is the “winding” number.

Then $x = \bar{x}(\tau) + q\sigma$, where $q = Rw$ is quantized “winding momentum” (cf. particle in a box).

Solution (2.101) has an additional symmetry (“ T -duality”) under $p \leftrightarrow q, \tau \leftrightarrow \sigma$.

Consider next the Lorentz rotations $\delta x^\mu = \varepsilon^\mu{}_\nu x^\nu$. The conserved current (angular momentum density) reads

$$j^a = \frac{\partial \mathcal{L}}{\partial \partial_a x^\mu} \varepsilon^\mu{}_\nu x^\nu = \frac{1}{2} j_{\mu\nu}^a \varepsilon^{\mu\nu}, \quad (2.102)$$

$$j_a^{\mu\nu}(\tau, \sigma) = T(x^\mu \partial_a x^\nu - x^\nu \partial_a x^\mu) \quad (2.103)$$

The corresponding conserved charge is

$$J^{\mu\nu} = \int d\sigma j^{0\mu\nu} = T \int d\sigma (\dot{x}^\mu x^\nu - \dot{x}^\nu x^\mu). \quad (2.104)$$

By construction, on the equations of motion one has

$$\frac{d}{d\tau} J_{\mu\nu} = 0. \quad (2.105)$$

Using (2.87), we can rewrite (2.104) as

$$J^{\mu\nu} = T \int d\sigma (\dot{\tilde{x}}^\mu \tilde{x}^\nu - \dot{\tilde{x}}^\nu \tilde{x}^\mu) + T \int d\sigma (\dot{\tilde{x}}^\mu \tilde{x}^\nu - \dot{\tilde{x}}^\nu \tilde{x}^\mu) = I^{\mu\nu} + S^{\mu\nu}. \quad (2.106)$$

Cross-terms here vanish due to (2.88) and (2.89). $I^{\mu\nu}$ is the orbital angular momentum and $S^{\mu\nu}$ is the internal one (i.e. the spin). Using (2.97) we can rewrite the orbital moment as

$$I^{\mu\nu} = P^\mu \tilde{x}^\nu - P^\nu \tilde{x}^\mu. \quad (2.107)$$

Using (2.97) we have

$$\frac{d}{d\tau} I^{\mu\nu} = \frac{d}{d\tau} (P^\mu \tilde{x}^\nu - P^\nu \tilde{x}^\mu) = (P^\mu \dot{\tilde{x}}^\nu - P^\nu \dot{\tilde{x}}^\mu) = 0.$$

With the help of the equations of motion we get also

$$\begin{aligned} \frac{d}{d\tau} S^{\mu\nu} &= T \int d\sigma (\ddot{\tilde{x}}^\mu \tilde{x}^\nu - \ddot{\tilde{x}}^\nu \tilde{x}^\mu) = T \int d\sigma (\tilde{x}''^\mu \tilde{x}^\nu - \tilde{x}''^\nu \tilde{x}^\mu) \\ &= T \int d\sigma \frac{\partial}{\partial \sigma} (\tilde{x}'^\mu \tilde{x}^\nu - \tilde{x}'^\nu \tilde{x}^\mu) = T (\tilde{x}'^\mu \tilde{x}^\nu - \tilde{x}'^\nu \tilde{x}^\mu) \Big|_0^L = 0. \end{aligned}$$

2.10 Rotating string solution

One simple solution is provided by a folded closed string rotating in the (x^1, x^2) -plane with its center of mass at rest

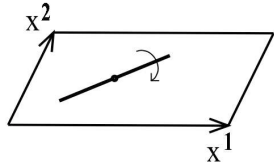


Fig.16: Rotating string

$$\begin{aligned} x^0 &= p^0 \tau \\ x^1 &= r(\sigma) \cos \phi(\tau) \\ x^2 &= r(\sigma) \sin \phi(\tau), \\ r(\phi) &= a \sin \omega \sigma, \quad \phi(\tau) = \omega \tau \end{aligned}$$

This solves $\ddot{x} - x'' = 0$ with a being an arbitrary constant. From the periodicity condition it follows that w is integer, i.e. $\omega = 0, \pm 1, \pm 2, \dots$. The constraint $\dot{x}^2 + x'^2 = 0$ yields $p^0 = a\omega$, and the constraint $\dot{x}x' = 0$ is satisfied automatically.

The Lagrangian of the string is

$$\mathcal{L} = -\frac{1}{2} T \partial_a x^\mu \partial^a x_\mu$$

so that the energy related the time-translation symmetry $x^0 \rightarrow x^0 + \epsilon$ is (we fix $L = 2\pi$)

$$E = P^0 = T \int_0^{2\pi} d\sigma \frac{\partial \mathcal{L}}{\partial \dot{x}^0} = T \int_0^{2\pi} d\sigma \dot{x}^0 = \frac{p^0}{\alpha'} = \frac{a\omega}{\alpha'}, \quad (2.108)$$

where $\alpha' = \frac{1}{2\pi T}$.

The spin is related to the rotation symmetry $\varphi \rightarrow \varphi + \epsilon$

$$S = T \int_0^{2\pi} d\sigma \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = T \int_0^{2\pi} d\sigma r^2(\sigma) \dot{\varphi} = T a^2 \omega \int_0^{2\pi} d\sigma \sin^2 \omega \sigma = \frac{a^2 \omega}{2\alpha'}. \quad (2.109)$$

Combining (2.108) and (2.109) we obtain relation between the energy and the spin

$$\alpha' E^2 = 2\omega S, \quad w = 1, 2, \dots \quad (2.110)$$

The configurations with the lowest energy for a given spin (or smallest slope on $E^2(S)$ plot) with $\omega = 1$ belong to the leading Regge trajectory. For them $\alpha' E^2 = 2S$ or recalling that $L = 2\pi$ we have

$$E^2 = 2T L S. \quad (2.111)$$

We can generalize the above rotating solution by adding motion to its center of mass:

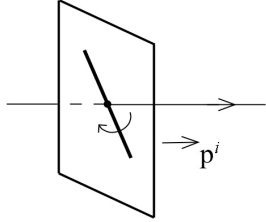


Fig.17: Rotating string with moving c.o.m.

$$\begin{aligned} x^0 &= p^0 \tau \\ x^1 &= x^1(\tau, \sigma), \quad x^2 = x^2(\tau, \sigma) \\ x^i &= x_0^i + p^i \tau, \quad i = 3, 4, \dots \end{aligned}$$

The constraint $\dot{x}^2 + x'^2 = 0$ then gives $p_0^2 - p_i^2 = a^2 \omega^2$. As in the case of the rotating string at rest we can find the expressions for the energy and the spin (2.110)

$$E^2 = P_i^2 + \frac{2\omega}{\alpha'} S, \quad P_i = \frac{p_i}{\alpha'}. \quad (2.112)$$

2.11 Orthogonal gauge, conformal reparametrizations and the light-cone gauge

In the orthogonal gauge the string action has the form

$$I = -\frac{1}{2} T \int d^2 \xi \partial_+ x^\mu \partial_- x_\mu. \quad (2.113)$$

The equations of motion and the constraints read

$$\partial_+ \partial_- x^\mu = 0, \quad \partial_\pm x^\mu \partial_\pm x_\mu = 0. \quad (2.114)$$

These are invariant under the residual conformal symmetry which is a subgroup of reparametrizations $\xi'^a = F_a(\xi^1, \xi^2)$ ($\xi^\pm = \tau \pm \sigma$)

$$\xi^{+'} = F_+(\xi^+), \quad \xi^{-'} = F_-(\xi^-). \quad (2.115)$$

Under such transformations the 2d metric changes by a conformal factor

$$ds^{2'} = -d\xi^{+'} d\xi^{-'} = -\frac{dF_+}{d\xi^+} \frac{dF_-}{d\xi^-} d\xi^+ d\xi^- \quad (2.116)$$

i.e. the orthogonal-gauge condition is preserved by such conformal reparametrizations.

One can choose a special parametrization by using this additional invariance, i.e. on the equations of motion for $x^\pm$ one can impose an additional gauge condition – the “light-cone gauge”

$$x^+(\tau, \sigma) = \bar{x}^+(\tau) = x_0^+ + p^+ \tau, \quad x^+ \equiv x^0 + x^{D-1}. \quad (2.117)$$

That means that $\tilde{x}^+(\tau, \sigma) = 0$, i.e. there is no oscillations in x^+ . Indeed, since

$$\partial_+ \partial_- x^+ = 0 \quad \rightarrow \quad x^+ = f_1^+(\tau + \sigma) + f_2^-(\tau - \sigma),$$

and one can use conformal reparametrizations to choose these functions so that $x^+ = \frac{1}{2}p(\tau + \sigma) + \frac{1}{2}p(\tau - \sigma) = p\tau$, where we also set $x_0^+ = 0$ by a shift.

This is a “physical gauge”, in which only the transverse oscillations x^i ($i = 1, 2, \dots, D-2$) are dynamical. Indeed, $\tilde{x}^-$ is then determined from the orthogonal gauge constraints

$$\partial_\pm x^\mu \partial_\pm x_\mu = 0 \quad \rightarrow \quad -\partial_\pm x^+ \partial_\pm x^- + \partial_\pm x^i \partial_\pm x^i = 0, \quad (2.118)$$

i.e.

$$\frac{1}{2}p^+ \partial_\pm x^- = \partial_\pm x^i \partial_\pm x^i, \quad (2.119)$$

which is a linear equation on x^- determining it in terms of x^i .

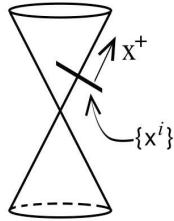


Fig.19: Light cone

We end up with

$$\begin{cases} x^+ = \bar{x}^+(\tau) \\ x^- = \bar{x}^-(\tau) + F[x^i(\tau, \sigma)] \\ x^i = \bar{x}^i(\tau) + \tilde{x}_L^i(\tau + \sigma) + \tilde{x}_R^i(\tau - \sigma) \end{cases}$$

x^i = dynamical degrees of freedom.

This gauge where x^i are dynamical is analogous to the light-cone gauge in quantum electrodynamics ($A_+ = A_0 + A_3 = 0$) where A_i ($i = 1, 2$) are dynamical degrees of freedom.

3 General solution of string equations in the orthogonal gauge

Let us use uniform notation for open/closed case: $\sigma = (0, \pi)$, i.e. choose $L = \pi$
Boundary conditions:

$$\begin{aligned} x(\tau, \sigma) &= x(\tau, \sigma + \pi) && \text{closed string} \\ \partial_\sigma x|_{\sigma=0} &= \partial_\sigma x|_{\sigma=\pi} = 0 && \text{open string} \end{aligned}$$

Equations of motion and constraints read

$$\partial_+ \partial_- x^\mu = 0, \quad \partial_\pm x^\mu \partial_\pm x_\mu = 0. \quad (3.1)$$

We set as before

$$x(\tau, \sigma) = x_0^\mu + p^\mu \tau + \tilde{x}^\mu(\tau, \sigma). \quad (3.2)$$

Then $\tilde{x}^\mu$ is given by Fourier mode expansion.

Open string:

Boundary conditions $\tilde{x}'^\mu|_{\sigma=0,\pi} = 0$ are satisfied by

$$\tilde{x}^\mu = \sum_{n \neq 0}^\infty a_n^\mu e^{-in\tau} \cos n\sigma \quad (3.3)$$

Reality condition of x^μ implies: $(a_n^\mu)^* = a_{-n}^\mu$. The momentum is

$$P^\mu = T \int_0^\pi d\sigma \dot{x}^\mu = T\pi p^\mu = \frac{1}{2\alpha'} p^\mu \quad (3.4)$$

The angular momentum (=orbital+spin) is

$$J^{\mu\nu} = I^{\mu\nu} + S^{\mu\nu} = \bar{x}^\mu p^\nu - \bar{x}^\nu p^\mu + T \int_0^\pi d\sigma (\tilde{x}^\mu \dot{\tilde{x}}^\nu - \tilde{x}^\nu \dot{\tilde{x}}^\mu) \quad (3.5)$$

Rescaling

$$a_n^\mu = i \frac{\sqrt{2\alpha'}}{n} \alpha_n^\mu$$

we get

$$\tilde{x}^\mu = i\sqrt{2\alpha'} \sum_{n \neq 0}^{\infty} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma. \quad (3.6)$$

Closed string:

Here $x(\sigma) = x(\sigma + \pi)$ and we can represent the oscillating part of the solution as a sum of the left-moving and right-moving waves $\tilde{x}^\mu(\tau, \sigma) = \tilde{x}_L^\mu(\xi^-) + \tilde{x}_R^\mu(\xi^+)$ ($\xi^\pm = \tau \pm \sigma$), i.e.

$$x^\mu(\tau, \sigma) = x_R^\mu(\xi^-) + x_L^\mu(\xi^+), \quad (3.7)$$

where

$$x_R^\mu(\xi^-) = \frac{1}{2}x_0^\mu + \frac{1}{2}\bar{p}^\mu\xi^- + \sum_{n \neq 0}^{\infty} a_n^\mu e^{-2in\xi^-} \quad (3.8)$$

$$x_L^\mu(\xi^+) = \frac{1}{2}x_0^\mu + \frac{1}{2}\bar{p}^\mu\xi^+ + \sum_{n \neq 0}^{\infty} \tilde{a}_n^\mu e^{-2in\xi^+} \quad (3.9)$$

Reality condition implies: $(a_n^\mu)^* = a_{-n}^\mu$, $(\tilde{a}_n^\mu)^* = \tilde{a}_{-n}^\mu$.

After the rescaling $a \rightarrow i\sqrt{\frac{\alpha'}{2}} \frac{\alpha_n}{n}$, $\tilde{a} \rightarrow i\sqrt{\frac{\alpha'}{2}} \frac{\tilde{\alpha}_n}{n}$, we get

$$x^\mu = x_0^\mu + 2\alpha' P^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0}^{\infty} \frac{\alpha_n^\mu}{n} e^{-2in(\tau-\sigma)} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0}^{\infty} \frac{\tilde{\alpha}_n^\mu}{n} e^{-2in(\tau+\sigma)} \quad (3.10)$$

The constraints imply the vanishing of the energy-momentum tensor $T_{\pm\pm} = (\partial_\pm x)^2 = 0$. Let us introduce its Fourier components. For closed string we get

$$L_n = \frac{T}{2} \int d\xi^- e^{-2in\xi^-} T_{--}(\xi^-) \quad (3.11)$$

$$\bar{L}_n = \frac{T}{2} \int d\xi^+ e^{-2in\xi^+} T_{++}(\xi^+) \quad (3.12)$$

where we integrate over σ in the range $(0, \pi)$ at $\tau = 0$. For the open string

$$L_n = T \int_0^\pi d\sigma^- (e^{in\sigma} T_{++} + e^{-in\sigma} T_{--}) = \frac{T}{4} \int_{-\pi}^\pi d\sigma e^{in\sigma} (\dot{x} + x')^2. \quad (3.13)$$

The substitution of the solutions (3.3) and (3.8) into these integrals yields

$$L_n = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{n-k}^\mu \alpha_{k\mu}, \quad \bar{L}_n = \frac{1}{2} \sum_{k=-\infty}^{\infty} \tilde{\alpha}_{n-k}^\mu \tilde{\alpha}_{k\mu} \quad (3.14)$$

for the closed string and

$$L_n = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{n-k}^{\mu} \alpha_{k\mu} \quad (3.15)$$

for the open string. Here we introduced the following definition:

$$\tilde{\alpha}_0^{\mu} = \alpha_0^{\mu} = \sqrt{\frac{\alpha'}{2}} p^{\mu} \quad \text{closed string} \quad (3.16)$$

$$\alpha_0^{\mu} = \sqrt{2\alpha'} p^{\mu} \quad \text{open string} \quad (3.17)$$

The angular momentum is found to be

$$J^{\mu\nu} = I^{\mu\nu} + S^{\mu\nu}(\alpha) + S^{\mu\nu}(\tilde{\alpha}), \quad (3.18)$$

where

$$I^{\mu\nu} = \bar{x}^{\mu} p^{\nu} - \bar{x}^{\nu} p^{\mu},$$

$$S^{\mu\nu}(\alpha) = i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^{\mu} \alpha_n^{\nu} - \alpha_{-n}^{\nu} \alpha_n^{\mu})$$

The string Hamiltonian is

$$H = T \int_0^{\pi} d\sigma (\dot{x}p - \mathcal{L}) = \frac{T}{2} \int_0^{\pi} d\sigma (\dot{x}^2 + x'^2) \quad (3.19)$$

i.e.

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu} = L_0 \quad \text{open string} \quad (3.20)$$

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} (\alpha_{-n}^{\mu} \alpha_{n\mu} + \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_{n\mu}) = L_0 + \bar{L}_0 \quad \text{closed string} \quad (3.21)$$

From (3.16), (3.20) and mass condition $p^2 = -M^2$ it follows that

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu} \quad \text{open string} \quad (3.22)$$

$$\alpha' M^2 = 2 \sum_{n=1}^{\infty} (\alpha_{-n}^{\mu} \alpha_{n\mu} + \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_{n\mu}) \quad \text{closed string} \quad (3.23)$$

Let us introduce the Poisson brackets

$$\{x^{\mu}(\sigma, \tau), x^{\nu}(\sigma', \tau)\} = \{\dot{x}^{\mu}, \dot{x}^{\nu}\} = 0, \quad (3.24)$$

$$\{\dot{x}^{\mu}(\sigma, \tau), x^{\nu}(\sigma', \tau)\} = T^{-1} \delta(\sigma - \sigma') \eta^{\mu\nu}, \quad (3.25)$$

Using the δ -function representation

$$\delta(\sigma) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{2in\sigma}$$

one can check that

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = im\delta_{m+n,0}\eta^{\mu\nu}, \quad (3.26)$$

$$\{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0. \quad (3.27)$$

For the zero modes we get $\{p^\mu, \bar{x}^\nu\} = \eta^{\mu\nu}$.

One finds then that

$$\{L_m, L_n\} = i(m-n)L_{m+n}, \quad (3.28)$$

$$\{\bar{L}_m, \bar{L}_n\} = i(m-n)\bar{L}_{m+n}, \quad \{L_m, \bar{L}_n\} = 0, \quad (3.29)$$

i.e. the constraints are in involution forming the Virasoro algebra.

From (3.14) and (3.15) it follows that

$$(L_n)^* = L_{-n}, \quad (\bar{L}_n)^* = \bar{L}_{-n}. \quad (3.30)$$

4 Covariant quantization of free string

4.1 Approaches to quantization

I. Canonical operator approach

Here one starts with promoting classical canonical variables to operators and imposing standard commutation relations.

In the Lorentz-covariant approach one uses covariant orthogonal gauge and imposes constraints on states. The Weyl symmetry is broken at the quantum level unless dimension $D = D_{crit} = 26$ (or $D = 10$ for superstrings). In this approach the theory is manifestly Lorentz-invariant, but there are no ghosts and the theory is unitary only in $D = D_{crit}$.

In the light cone (“physical”) gauge approach the constraints are first solved explicitly at the classical level, and then only the remaining transverse modes are quantized. Here the theory is manifestly unitary but is Lorentz-invariant only in $D = D_{crit}$.

II. Path integral approach

Here one starts with the path integral defined by the string action

$$\langle \dots \rangle = \int Dg_{ab} \int Dx^\mu e^{iS[x,g]} \dots$$

The 2d metric is assumed to be non-dynamical (the Weyl symmetry is assumed to be an exact symmetry of the quantum theory, which happens to be true only if $D = D_{crit}$), and the integral over the continuous (local) modes of the metric is removed by the gauge fixing.

The observables here are correlation functions of “vertex operators” (corresponding to string states) defining the scattering amplitudes of particular string modes.

4.2 Covariant operator quantization

We promote canonical center-of-mass variables (x_0, p) to operators $(\hat{x}_0, \hat{p})$, and do the same for the oscillation modes: $\alpha \rightarrow \hat{\alpha}, \tilde{\alpha} \rightarrow \hat{\tilde{\alpha}}$.

The Poisson brackets $\{, \}$ are replaced by the quantum commutator $-i[,]$.

The string state $(p, \alpha, \tilde{\alpha})$ then corresponds to a quantum state vector $|\psi\rangle$ in Hilbert space.

The reality condition $\alpha_n = \alpha_{-n}^*$ becomes $\hat{\alpha}_n = \hat{\alpha}_{-n}^\dagger$.

Then we get $[\hat{x}_0^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}$, where $\hat{x}^\mu = \hat{x}_0^\mu + 2\alpha' \hat{p}^\mu + \dots$ and also (below we shall often omit “hats” on the operators)

$$[\alpha_m^\mu, \alpha_{-m}^\nu] = m \eta^{\mu\nu}, \quad [\alpha_m^\mu, \alpha_n^\nu] = 0, \quad m \neq -n \quad (4.1)$$

The “Fock vacuum” $|0, p\rangle$ satisfies

$$\hat{p}|0, p\rangle = p|0, p\rangle. \quad (4.2)$$

The vacuum is annihilated by α_m^μ , $m > 0$

$$a_m^\mu |0, p\rangle = 0 \quad (m > 0), \quad a_m^\mu \equiv \frac{1}{\sqrt{m}} \alpha_m^\mu \quad (4.3)$$

The general string state vector is

$$|\psi\rangle = (a_{n_1}^\dagger)^{k_1} \dots (a_{n_l}^\dagger)^{k_l} |0, p\rangle \quad (4.4)$$

Not all states are physical. They must satisfy the constraints and have positive norm. But

$$[\alpha_n^0, \alpha_n^{0\dagger}] = -1, \quad n > 0,$$

implies existence of negative norm states

$$||\alpha_n^0 |0\rangle|| = -1$$

These can be ruled out by imposing the quantum version of the constraints.

4.2.1 Open string

Here the Virasoro operators are

$$L_0 = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu} \quad (4.5)$$

$$L_m = L_{-m}^{\dagger} = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_{m-n}^{\mu} \alpha_{n\mu} , \quad m > 0. \quad (4.6)$$

In the classical theory the constraints are $L_0 = 0$, $L_m = 0$. In the quantum theory L_0 and L_m become operators and the problem of ordering of the operators that enter them arises.

The normal ordering is automatic for $m \neq 0$ since $m - n \neq -n$

$$L_m \rightarrow \hat{L}_m = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_{m-n}^{\mu} \alpha_{n\mu} , \quad m \neq 0. \quad (4.7)$$

For $m = 0$ we have in general ($c_1 + c_2 = 1$)

$$\begin{aligned} L_0 \rightarrow \hat{L}_0 &= \alpha' p^2 + \sum_{n=1}^{\infty} (c_1 \alpha_{-n}^{\mu} \alpha_{n\mu} + c_2 \alpha_n^{\mu} \alpha_{-n\mu}) \\ &= \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu} + c_2 \sum_{n=1}^{\infty} [\alpha_n^{\mu}, \alpha_{-n\mu}] \end{aligned}$$

Then

$$\hat{L}_0 =: \hat{L}_0 : + a , \quad a = c_2 D \sum_{n=1}^{\infty} n. \quad (4.8)$$

It turns out that for consistency of the theory one is to choose $a = -\frac{D}{24}$. This value is found if one uses the ζ -function regularisation and chooses also $c_1 = c_2 = \frac{1}{2}$. Consider the series

$$\begin{aligned} I_0(\varepsilon) &= \sum_{n=1}^{\infty} e^{-\varepsilon n} = \frac{1}{e^{\varepsilon} - 1} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} - \frac{1}{2} + O(\varepsilon) \\ I_1(\varepsilon) &= -\frac{d}{d\varepsilon} I_0 = \sum_{n=1}^{\infty} n e^{-\varepsilon n} = \frac{e^{\varepsilon}}{(e^{\varepsilon} - 1)^2} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} - \frac{1}{12} + O(\varepsilon) \\ I_{-1}(\varepsilon) &= -\int d\varepsilon I_0(\varepsilon) = \sum_{n=1}^{\infty} \frac{1}{n} e^{-\varepsilon n} = -\ln(1 - e^{-\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} -\ln \varepsilon + O(\varepsilon) \end{aligned}$$

The ζ -function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C} \quad (4.9)$$

and then at $s = 0, 1, -1, \dots$ by an analytic continuation. This gives the same values as finite parts in the above series

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12}, \quad \zeta(1) = \infty \quad (4.10)$$

With this prescription

$$a = \frac{1}{2}D\zeta(-1) = -\frac{D}{24}. \quad (4.11)$$

In fact, the true value of the normal ordering constant a is not $-\frac{D}{24}$ but $a = -\frac{D-2}{24}$: the additional -2 contribution to D comes from the “ghosts” of the conformal gauge (in the light-cone gauge, only $D - 2$ transverse modes x^i contribute).

The quantum version of the constraints is then

$$(: L_0 : + a)|\psi\rangle = 0, \quad (4.12)$$

$$L_m|\psi\rangle = 0, \quad m > 0 \quad (4.13)$$

where

$$: L_0 := \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu}, \quad L_m = \sum_{n=0}^{\infty} \alpha_{m-n}^{\mu} \alpha_{n\mu} \quad (4.14)$$

The mass shell condition $p^2 = -M^2$ then gives the expression for the mass operator

$$\alpha' M^2 |\psi\rangle = (a + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu}) |\psi\rangle \quad (4.15)$$

In particular, the value a thus determines the mass of the ground state

$$\alpha' M^2 |0, p\rangle = a |0, p\rangle, \quad (4.16)$$

which is tachyonic for $D = 26$: $a = -\frac{D-2}{24} = -1$. In the supersymmetric string case the ground state turns out to be massless.

For the Virasoro algebra one finds

$$\text{Classical:} \quad \{L_m, L_n\} = (m - n)L_{m+n} \quad (4.17)$$

$$\text{Quantum:} \quad [L_m, L_n] = (m - n)L_{m+n} + \frac{D}{12}(m^2 - 1)m \delta_{m+n,0} \quad (4.18)$$

where the quantum algebra turns out to contain a central term. When L_m are modified by the ghost contributions the coefficient D in the central term is shifted to $D - 2$.

Note that for $n = -m$ we get

$$[L_m, L_{-m}] = 2mL_0 + \frac{D}{12}(m^3 - m) \quad (4.19)$$

so that in view of $(L_0 + a)|\psi\rangle = 0$

$$[L_m, L_{-m}]|\psi\rangle = 2m\left[L_0 + \frac{D}{24}(m^2 - 1)\right]|\psi\rangle \neq 0, \quad (4.20)$$

which is in contradiction with $L_m|\psi\rangle = 0$ for $m \neq 0$. To avoid this contradiction it is sufficient to impose the constraints “on average”, i.e. to demand that matrix elements of L_m with $m \neq 0$ should vanish:

$$\langle \psi' | L_m | \psi \rangle = 0, \quad m \neq 0 \quad (4.21)$$

The spectrum is then described in terms of the oscillator states:

$$\hat{L}_0 = L_0 + a = \alpha' p^2 + \sum_{n=1}^{\infty} n N_n + a, \quad (4.22)$$

$$N_n \equiv \frac{1}{n} \alpha_{\mu n}^\dagger \alpha_n^\mu = a_{\mu n}^\dagger a_n^\mu, \quad [a_n^\mu, a_n^{\nu\dagger}] = \eta^{\mu\nu}. \quad (4.23)$$

N_n has the following properties

$$N_n |0\rangle = 0, \quad N_n a_n^{\nu\dagger} |0\rangle = a_n^{\nu\dagger} |0\rangle. \quad (4.24)$$

For generic oscillator state

$$|\psi\rangle = (a_{n_1}^\dagger)^{i_1} \dots (a_{n_k}^\dagger)^{i_k} |0, p\rangle, \quad \hat{p}^\mu |0, p\rangle = p^\mu |0, p\rangle \quad (4.25)$$

where $(a_{n_1}^\dagger)^{i_1}$ stands for $a_{n_1}^{\mu_1\dagger} \dots a_{n_1}^{\mu_{i_1}\dagger}$ we can define the level number as

$$\ell = i_1 n_1 + \dots + i_k n_k \quad (4.26)$$

Then

$$\sum_{n=1}^{\infty} n N_n |\psi\rangle = (i_1 n_1 + \dots + i_k n_k) |\psi\rangle = \ell |\psi\rangle \quad (4.27)$$

and thus the mass of this state is determined by ($T = \frac{1}{2\pi\alpha'}$)

$$M^2 |\psi\rangle = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} n N_n + a \right) |\psi\rangle = 2\pi T (\ell + a) |\psi\rangle. \quad (4.28)$$

In the open string case we may call the physical states those that satisfy the condition

$$L_m|\psi\rangle = 0 \quad \text{for } m \geq 1. \quad (4.29)$$

Let us define the “true physical states” as those that have positive norm

$$\langle \psi | \psi \rangle > 0. \quad (4.30)$$

Other physical states that have zero norm are called “null states”. One can prove the “no ghost” theorem: iff $D = 26$ all physical states have non-negative norm.

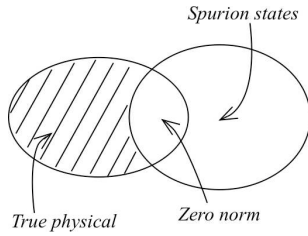


Fig.20: String states

One can define spurious states as those obtained by acting by L_{-m} , i.e. $|\chi\rangle = L_{-m}|\phi\rangle$ for $m > 0$. Such states orthogonal to physical since

$$L_m|\psi\rangle = 0 \quad \rightarrow \quad \langle \psi | L_m^\dagger = \langle \psi | L_{-m} = 0$$

$$\langle \psi | L_{-m}|\phi\rangle = 0 \quad \rightarrow \quad \langle \psi | \chi \rangle = 0.$$

Null states may be both physical and spurious; true physical states are then non-spurious ones, i.e. physical ones that have positive norm.

4.2.2 Closed string

Here we get two independent sets of creation-annihilation operators α_m^μ and $\tilde{\alpha}_m^\mu$. The normal ordering ambiguity in going from classical to quantum theory implies

$$L_0 \rightarrow L_0 + a, \quad \bar{L}_0 \rightarrow \bar{L}_0 + a.$$

$$\hat{L}_0 + \hat{\bar{L}}_0 = \frac{\alpha' p^2}{2} + \sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_{n\mu} + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu}) + 2a, \quad (4.31)$$

$$\hat{L}_m = \sum_{n=-\infty}^{\infty} \alpha_{m-n}^\mu \alpha_{n\mu}, \quad \hat{\bar{L}}_m = \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}^\mu \tilde{\alpha}_{n\mu} \quad (4.32)$$

$$a_{\text{closed}} = 2a_{\text{open}} = 2a = -\frac{D}{12}, \quad (4.33)$$

where D is shifted to $D - 2$ after one accounts for the conformal gauge ghosts or uses the light-cone gauge.

Here one gets two copies of the Virasoro algebra – for L_n and $\bar{L}_n$.

The vacuum state is defined by

$$\hat{p}^\mu |0, p\rangle = p^\mu |0, p\rangle, \quad \alpha_n^\mu |0, p\rangle = 0, \quad \tilde{\alpha}_n^\mu |0, p\rangle = 0, \quad n > 0 \quad (4.34)$$

and

$$|\psi\rangle = (\alpha_{n_1}^\dagger)^{i_1} \dots (\alpha_{n_k}^\dagger)^{i_k} (\tilde{\alpha}_{m_1}^\dagger)^{j_1} \dots (\tilde{\alpha}_{m_s}^\dagger)^{j_s} |0\rangle \quad (4.35)$$

form an infinite set of states in Fock space. The level numbers are defined by

$$\ell = i_1 n_1 + \dots + i_k n_k, \quad \bar{\ell} = m_1 j_1 + \dots + m_s j_s. \quad (4.36)$$

The constraints in classical theory are $L_0 = 0$, $\bar{L}_0 = 0$ and $L_m = 0$, $\bar{L}_m = 0$ for $m \neq 0$. In quantum theory we need to impose them “on average” to avoid contradiction with the centrally extended Virasoro algebra; that means we need to assume that

$$(L_0 + a)|\psi\rangle = 0, \quad (\bar{L}_0 + a)|\psi\rangle = 0, \quad (4.37)$$

$$L_m|\psi\rangle = 0, \quad \bar{L}_m|\psi\rangle = 0 \quad \text{for } m > 0. \quad (4.38)$$

Equivalently, (4.37) reads

$$(L_0 + \bar{L}_0 + 2a)|\psi\rangle = 0, \quad (4.39)$$

$$(L_0 - \bar{L}_0)|\psi\rangle = 0. \quad (4.40)$$

Then the constraints are satisfied for the expectation values

$$\langle \psi' | L_m | \psi \rangle = 0, \quad \langle \psi' | \bar{L}_m | \psi \rangle = 0, \quad m \neq 0. \quad (4.41)$$

Equation (4.39) determines the mass spectrum and (4.40) is the “level matching” condition.

From (4.39) it follows that

$$\left(\frac{\alpha' p^2}{2} + \sum_{n=1}^{\infty} n(N_n + \tilde{N}_n) + 2a \right) |\psi\rangle = 0, \quad (4.42)$$

where

$$N_n = a_n^{\mu\dagger} a_{\mu n} = \frac{1}{n} \alpha_n^{\mu\dagger} \alpha_{\mu n}, \quad \tilde{N}_n = \tilde{a}_n^{\mu\dagger} \tilde{a}_{\mu n} = \frac{1}{n} \tilde{\alpha}_n^{\mu\dagger} \tilde{\alpha}_{\mu n}. \quad (4.43)$$

Then the closed-string states have masses

$$M^2 = \frac{2}{\alpha'} (\ell + \bar{\ell} + 2a). \quad (4.44)$$

Eq. (4.40) gives

$$\sum_{n=1}^{\infty} n(N_n - \tilde{N}_n) |\psi\rangle = 0, \quad \text{i.e.} \quad \ell = \bar{\ell} \quad (4.45)$$

so that we get

$$M^2 = \frac{4}{\alpha'}(\ell + a). \quad (4.46)$$

The mass of the ground state is thus

$$M_0^2 = \frac{4}{\alpha'}a. \quad (4.47)$$

As in the open string case with $D = 26$ here we get $2a = -\frac{D-2}{12} = -2$, so that the ground state is tachyonic.

The structure of the physical state space here is similar to the one in the open string case.

5 Light cone gauge description of the free string spectrum

The equations of motion and the constraints in the orthogonal gauge

$$\ddot{x} - x'' = 0, \quad \dot{x}^2 + x'^2 = 0, \quad \dot{x}x' = 0. \quad (5.1)$$

have residual conformal reparametrization symmetry

$$\xi^+ \rightarrow f(\xi^+), \quad \xi^- \rightarrow \tilde{f}(\xi^-), \quad \sigma^\pm = \tau \pm \sigma \quad (5.2)$$

As was already mentioned above, it can be fixed by imposing an additional “light cone gauge” condition

$$x^+(\sigma, \tau) = \bar{x}^+(\tau) = x_0^+ + 2\alpha' p^+ \tau, \quad x^\pm = x^0 \pm x^{D-1}, \quad (5.3)$$

implying that there is no oscillator part in the classical solution for x^+ .

Then the constraints determine

$$x^-(\sigma, \tau) = x_0^- + 2\alpha' p^- \tau + F(p^+, x^i(\sigma, \tau)). \quad (5.4)$$

in terms of x^i , i.e. $D - 2$ independent “transverse” degrees of freedom.

5.1 Open string

Using constraint $\dot{x} \cdot x' = 0$ we get

$$0 = \dot{x}^\mu x'_\mu = -\frac{1}{2}(\dot{x}^+ x'^- + \dot{x}^- x'^+) + \dot{x}^i x'^i = -\alpha' p^+ + \dot{x}^i x'^i, \quad (5.5)$$

so that

$$x^- = \frac{1}{\alpha' p^+} \int d\sigma \dot{x}^i x'^i + h(\tau) \quad (5.6)$$

and thus

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'p^+}} \sum_{m=0}^{\infty} \alpha_{n-m}^i \alpha_m^i. \quad (5.7)$$

From $\dot{x}^2 + x'^2 = 0$ it follows that

$$2\alpha'p^+\dot{x}^- = \dot{x}^i \dot{x}^i + x'^i x'^i. \quad (5.8)$$

Here we quantize as independent ones the transverse oscillators only. The mass shell condition $-\alpha'p^2 = \alpha'M^2 = N = N_{\perp}$ is then determined by the transverse-mode oscillation number

$$N_{\perp} = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i. \quad (5.9)$$

In quantum theory

$$\hat{N}_{\perp} = : N_{\perp} : + a, \quad a = -\frac{D-2}{24} \quad (5.10)$$

If $D = 26$ then $a = -1$ and one can show that the resulting spectrum is consistent with the requirement of the Lorentz invariance of the theory.

Since the constraints are solved already, by acting on the Fock vacuum by the creation operators $\alpha_{-n}^i = (\alpha_n^i)^{\dagger}$ we get only the physical states

$$|\psi\rangle = \xi_{i_1 \dots i_m}(p) \alpha_{-n_1}^{i_1} \dots \alpha_{-n_m}^{i_m} |0, p\rangle \quad (5.11)$$

$$\alpha_{n>0}^i |0, p\rangle = 0, \quad \hat{p}^{\mu} |0, p\rangle = p^{\mu} |0, p\rangle, \quad [x_0^{\mu}, p^{\nu}] = i\eta^{\mu\nu} \quad (5.12)$$

The angular momentum $J^{\mu\nu} = I^{\mu\nu} + S^{\mu\nu}$ here is

$$I^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu}, \quad S^{\mu\nu} = -2i\alpha' \sum_n (\alpha_{-n}^{\mu} \alpha_n^{\nu} - \alpha_{-n}^{\nu} \alpha_n^{\mu}). \quad (5.13)$$

The Lorentz algebra is defined by

$$[J^{\mu\nu}, J^{\lambda\rho}] = \eta^{\mu\lambda} J^{\nu\rho} - \eta^{\nu\lambda} J^{\mu\rho} - \eta^{\mu\rho} J^{\nu\lambda} + \eta^{\nu\rho} J^{\mu\lambda}, \quad (5.14)$$

where

$$J^{\mu\nu} = (J^{ij}, J^{+i}, J^{-i}, J^{+-}).$$

The commutation relation $[J^{-i}, J^{-j}] = 0$ realized in terms of the quantum string oscillators turns out to be valid only if $D = 26$: the light cone gauge preserves Lorentz symmetry iff $D = 26$ (in superstring theory one finds this for $D = 10$).

The lowest-mass states in the spectrum are:

$$N_{\perp} = \ell = 0 : \quad \alpha'M^2 = a = -1$$

$$N_{\perp} = \ell = 1 : \quad \alpha' M^2 = 1 - 1 = 0$$

$$|\psi\rangle = \xi_i(p) \alpha_{-1}^i |0, p\rangle, \quad i = 1, \dots, D-2$$

$\xi_i(p)$ is a vector of $SO(D-2)$ having $D-2 = 24$ physical polarisations; this is consistent with Lorentz invariance since it is massless.

$$N_{\perp} = \ell = 2 : \quad \alpha' M^2 = 2 - 1 = 1 \text{ There are two possibilities}$$

$$|\psi\rangle = \xi_i(p) \alpha_{-2}^i |0, p\rangle \quad (5.15)$$

$$|\psi\rangle' = \xi_{ij}(p) \alpha_{-1}^i \alpha_{-1}^j |0, p\rangle \quad (5.16)$$

The total number of components is

$$D-2 + \frac{(D-2)(D-2+1)}{2} = \frac{D(D-1)}{2} - 1.$$

This is a dimension of $SO(D-1)$ representation (symmetric traceless tensor of $SO(D-1)$) as it should be for massive state in a Lorentz-invariant theory in D dimensions. Thus the $\ell = 2$ state is a spin-2 massive particle.

Higher levels are described by $SO(D-1)$ representations as well. Let us recall that the irreducible representations of $SO(r)$ are described by tensors $t_{m_1 \dots m_k}$, $m_i = 1, \dots, r$ which are symmetric traceless, or antisymmetric or have mixed symmetry and can be represented by the Young tableaux.

Higher level states in the light cone gauge spectrum are described by tensors of $SO(D-2)$ which combine into irreducible representations of $SO(D-1)$, i.e. by massive particles in D dimensions.

The maximal-spin state at level ℓ with $\alpha' M^2 = \ell - 1$ is

$$|\psi\rangle = \xi_{i_1 \dots i_{\ell}}(p) \alpha_{-1}^{i_1} \dots \alpha_{-1}^{i_{\ell}} |0, p\rangle, \quad (5.17)$$

where $\xi_{i_1 \dots i_{\ell}}$ is a symmetric tensor of $SO(D-2)$. To get the full state of spin ℓ one has to add lower tensors of $SO(D-2)$.

For instance, for $\ell = 3$:

$\xi_{ijk} \alpha_{-1}^i \alpha_{-1}^j \alpha_{-1}^k 0, p\rangle$	symmetric 3rd rank tensor
$\xi_{ij} \alpha_{-2}^i \alpha_{-1}^j 0, p\rangle$	symmetric + antisymmetric 2nd rank tensor
$\xi_i \alpha_{-3}^i 0, p\rangle$	vector

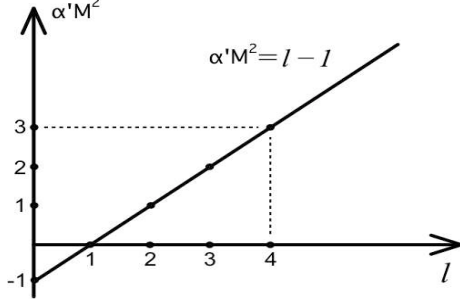


Fig.21: Regge trajectory

Leading Regge trajectory (highest spin states at given level):

$$J = \ell = \alpha(E) = \alpha(0) + \alpha' E^2 = 1 + \alpha' M^2$$

5.2 Closed string

Here the transverse oscillators are $(\alpha_n^i, \tilde{\alpha}_n^i)$; these are the only ones that remain after we set $\alpha_n^\dagger = 0$ and $\tilde{\alpha}_n^\dagger = 0$ (light cone gauge) and use the constraints to determine $\alpha_n^-, \tilde{\alpha}_n^-$ as functions of $\alpha_n^i, \tilde{\alpha}_n^i$. Then the relevant oscillator number operators are

$$N = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i, \quad \bar{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \quad (5.18)$$

The mass shell conditions are

$$(L_0 - 1)|\psi\rangle = 0, \quad (\bar{L}_0 - 1)|\psi\rangle = 0. \quad (5.19)$$

The physical states are

$$|\psi\rangle = \xi_{i_1 \dots i_n, j_1 \dots j_k}(p) \alpha_{-m_1}^{i_1} \dots \alpha_{-m_n}^{i_n} \tilde{\alpha}_{-s_1}^{j_1} \dots \tilde{\alpha}_{-s_k}^{j_k} |0, p\rangle, \quad (5.20)$$

For them

$$\alpha' M^2 = 2(\ell + \bar{\ell}) - 4, \quad \ell - \bar{\ell} = 0, \quad (5.21)$$

$$\ell = i_1 m_1 + \dots + i_n m_n, \quad \bar{\ell} = j_1 s_1 + \dots + j_k s_k \quad (5.22)$$

i.e. $\alpha' M^2 = 4(\ell - 1)$. We thus find:

$$\ell = 0: \quad \alpha' M^2 = -4 \text{ — scalar tachyon}$$

$$\ell = 1: \quad \alpha' M^2 = 0 \text{ — massless state}$$

This state $\xi_{ij}(p) \alpha_{-1}^i \tilde{\alpha}_{-1}^j |0, p\rangle$ can be split into irreps of $SO(D-2)$:

$$\xi_{ij} = \bar{\xi}_{(ij)} + \delta_{ij} \phi + \xi_{[ij]} \quad (5.23)$$

The symmetric traceless tensor $\bar{\xi}_{(ij)}$ represents spin-2 graviton which lies on the leading Regge trajectory. The second term ϕ is a scalar dilaton. The antisymmetric tensor corresponding to $\xi_{[ij]}$ is called the Kalb-Ramond field.

The graviton has $\frac{(D-2)(D-2+1)}{2} - 1 = \frac{D(D-3)}{2}$ components, and the Kalb-Ramond field has $\frac{(D-2)(D-3)}{2}$ components (in $D = 4$ the graviton has two degrees of freedom and the Kalb-Ramond field has one, i.e. is equivalent to a scalar).

Higher massive levels are described by $SO(D-2)$ tensors combined in irreps of $SO(D-1)$.

The superstring spectrum has similar structure with the ground state being massless (representing $D = 10$ supergravity states) and with fermions as well as bosons as physical states.

Many more details and various extensions can be found in the books listed below.

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Some books on string theory

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- [2] A. M. Polyakov, “Gauge fields and strings,” *Chur, Switzerland: Harwood (1987) 301p*.
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- [4] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string; Vol. 2: Superstring theory and beyond,” *Cambridge, UK: Univ. Pr. (1998) 402p; 531p*
- [5] B. Zwiebach, “A first course in string theory,” *Cambridge, UK: Univ. Pr. (2004) 558p*
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- [7] E. Kiritsis, “String theory in a nutshell,” *Princeton, USA: Univ. Pr. (2007) 588p*